Stochastic Quadratic Knapsack with Recourse

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Outline

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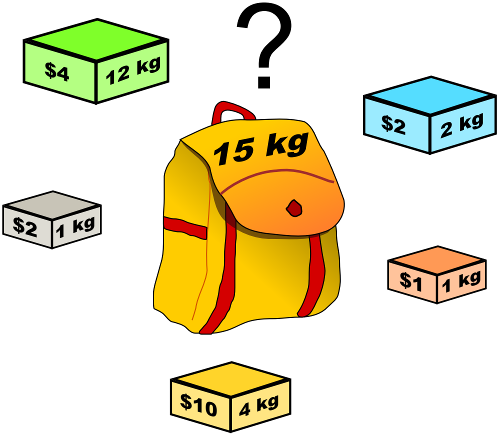
- A feasibility of selection of items is determined by a capacity restriction $\sum_{j=1}^{n} w_j x_j \leq d$.
- The objective is to select a subset of items such that the total profit of the selected items is maximized: $\max \sum_{j=1}^{n} c_j x_j$

Knapsack Problem



- The knapsack problem has been widely studied for the last decades
- (KP) is (weakly) NP-complete problem.
- Exact solutions, heuristics and approximations methods have been proposed.
- In the linear formulation of (KP), the profit of choosing an item is independent of the other items chosen.

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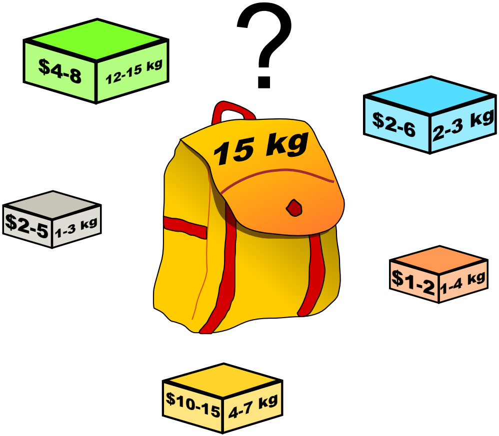
- Example: bread and butter, butter and knife.
- The Quadratic Knapsack problem (QKP) allows the formulation of such an interdependence.
- In the QKP, an item has a corresponding profit and an additional profit is redeemed if the item is selected with another one.
- In (QKP), the profit is defined by a matrix $C = (c_{ij})$ where c_{jj} is the return is item j is selected, and c_{ij} is the return if items i and j are selected $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j$.

Outline

What happens when we take uncertainty into account?



The Problem we Study.

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- We are interested in a two-stage decision.
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- Second period: more information is revealed, we take a corrective decision.
- For example, initial knapsack packing was prepared with some estimated customer demand. In the second period, more information is avaliable and we correct the decision by adding extra items and/or removing others.

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There are multiple stochastic aspects that can play a role:

- Uncertainty about the future: we have the possibility of recourse
- Incomplete information and risk taking: probability constraints
  - In the first stage
  - In the second stage
  - In both stages

Outline

First Stage Decision.

$$\max_x \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} x_i x_j + \mathbb{E}_\omega Q(u, \omega) \tag{1}$$

$$\sum_{i=1}^{N} w_i x_i \leq d \tag{2}$$

Second Stage Decision.

$$Q(u,\omega) = \max_{u,u^-} \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij}(\omega) u_i u_j - \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij}^{-}(\omega) u_i^{-} u_j^{-} \quad (3)$$

$$u_i \geq x_i - u_i^{-}, \; i = 1:n \quad (4)$$

$$u_i^{-} \leq x_i, \; i = 1:n \quad (5)$$

$$\sum_{i=1}^{N} v_i(\omega)(u_i + x_i - u_i^{-}) \leq h(\omega) \quad (6)$$

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- The risk of violating the capacity constraints is given by α_1
- The modified first stage capacity constraint is:

$$\mathbb{P}\left\{\sum_{i=1}^{N} w_i(\phi)x_i \leq d\right\} \geq (1-\alpha_1) \qquad (7)$$

In the Second Stage

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In the Second Stage

Similarly,

- We know the distribution of probability vector ψ, conditioned on ω.
- ψ represents the uncertainty on the weights of the items.
- The risk of violating the capacity constraints is given by α_2
- The modified second stage capacity constraint is:

$$\mathbb{P}\left\{\sum_{i=1}^{N} v_i(\omega,\psi)(u_i + x_i - u_i^-) \leq h(\omega,\psi) | \omega\right\} \geq (1-\alpha_2) \quad (8)$$

Outline

Base Model.

- First stage decision:

$$\max_{x} \left[\sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} x_i x_j + \sum_{k=1}^{K} p_k^{\omega} Q(u,k) \right] \tag{9}$$

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- Second stage decision:

$$Q(u,k) = \max_{u,u^-}\left[\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ijk}u_{ik}u_{jk} - \sum_{i=1}^{N}\sum_{j=1}^{N} b_{ijk}^{-}u_{ik}^{-}u_{jk}^{-}\right] \tag{10}$$

Base Model.

- Second stage capacity constraint:

$$\sum_{i=1}^{N} v_{ik}(u_{ik} + x_i - u_{ik}^{-}) \leq h_k \quad \forall k = 1 : K \tag{11}$$

Putting it Together.

$$\max_{x,u_{ik},u_{ik}^-} \left[\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}x_ix_j + \sum_{k=1}^{K} p_k^{\omega} \left(\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ijk}u_{ik}u_{jk} - \sum_{i=1}^{N}\sum_{j=1}^{N} b_{ijk}^{-}u_{ik}^{-}u_{jk}^{-} \right) \right]$$

$$\text{s.t.} \quad \sum_{i=1}^{N} w_ix_i \le d$$

$$u_{ik} \ge x_i - u_{ik}^{-}, \; i = 1:n, \; k = 1:K$$

$$u_{ik}^{-} \le x_i, \; i = 1:n, \; k = 1:K$$

$$\sum_{i=1}^{N} v_{ik}(u_{ik} + x_i - u_{ik}^{-}) \le h_k, \; k = 1:K \tag{12}$$

Probability Constraints.

- Suppose that the random vector ϕ is concentrated in the finite number of points $\phi_l, l = 1 : L$
- We note Γ a subset of scenarios of set $\{1, ..., L\}$
- Then constraint $\mathbb{P}\left\{\sum_{i=1}^{N} w_i(\phi)x_i \leq d\right\} \geq (1-\alpha_1)$ is equivalent to the pair:

$$\begin{cases} \displaystyle\sum_{i=1}^{N} w_{il}x_i \leq d_l, \;\; l \in \Gamma \\ \displaystyle\sum_{l\in\Gamma} p_l^{\phi} \geq 1-\alpha_1, \;\; k = 1 : K \end{cases}$$

Probability Constraints (cont.).

- We introduce binary variables y_l^{ϕ} to indicate if constraint l does not belong to Γ.

$$y_l^{\phi} = \begin{cases} 0 & \text{if} \quad l \in \Gamma \\ 1 & \text{otherwise} \end{cases}$$

- This yields the following deterministic equivalent constraints:

$$\begin{cases} \sum_{i=1}^{N} w_{il} x_i \leq d_l + M_l^{\phi} y_l^{\phi} \\ \sum_{l=1}^{L} p_l^{\phi} y_l^{\phi} \leq \alpha_1 \end{cases} \tag{13}$$

Probability Constraints (end).

- For the second stage probability constraints, we use the same reasoning
- This yields the following deterministic equivalent constraints:

$$\begin{cases} \sum_{i=1}^{N} v_{ikr}(u_{ik} + x_i - u_{ik}^{-}) \leq h_{kr} + M_k^{\psi} y_{kr}^{\psi}, \ r = 1 : R, \ k = 1 : K \\ \sum_{r=1}^{R} p_{kr}^{\psi} y_{kr}^{\psi} \leq \alpha_2, \ k = 1 : K \end{cases} \quad (14)$$

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- Small values of the parameters N, K, R and L lead to large combinatorial problems.
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  - Linear relaxation. The advantage is that it is simple to solve a linear program. However, the results are poor (large gap with the optimum)
  - SDP relaxations. The advantage is that it is proven to give better bounds than the LP when using additional valid constraints, however it is computationally more expensive (but still polynomial).
  - Interior point vs bundle method: these are two main ways to solve SDP problems, IP methods converge faster, but reach size limits faster than bundle methods.

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- We add fortet constraints $X_{ij} \leq x_i$, $X_{ij} \leq x_j$ and $x_i + x_j - X_{ij} \leq 1$.
- Finally, we relax the constraints $x_i \in \{0, 1\}$ into $0 \leq x_i \leq 1$.

Linear Relaxation.

$$\max_{x,u_{ik},u^-_{ik}} \left[\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}X_{ij} + \sum_{k=1}^{K} p_k^{\omega} \left(\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ijk}U_{ijk} - \sum_{i=1}^{N}\sum_{j=1}^{N} b^-_{ijk}U^-_{ijk} \right) \right]$$

$$\begin{aligned}
\text{s.t.}\quad & \sum_{i=1}^{N} w_i x_i \le d \\
& u_{ik} \ge x_i - u^-_{ik},\ i = 1:n,\ k = 1:K \\
& u^-_{ik} \le x_i,\ i = 1:n,\ k = 1:K \\
& \sum_{i=1}^{N} v_{ik}(u_{ik} + x_i - u^-_{ik}) \le h_k\ ,\ k = 1:K \\
& X_{ij} \le x_i,\ X_{ij} \le x_j,\ x_i + x_j - X_{ij} \le 1 \\
& U_{ijk} \le u_{ik},\ U_{ijk} \le u_{jk},\ u_{ik} + u_{jk} - U_{ijk} \le 1 \\
& U^-_{ijk} \le u^-_{ik},\ U^-_{ijk} \le u^-_{jk},\ u^-_{ik} + u^-_{jk} - U^-_{ijk} \le 1
\end{aligned} \tag{15}$$

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- In semidefinite programming, the problem is defined over the positive semidefinite cone
- We can add constraints that linear programming cannot model.
- Polynomial algorithms exist for semidefinite programming.

Semidefinite Relaxations (1).

- Let **X** be the matrix defined as:

$$\mathbf{X} = \begin{bmatrix} xx^t & x \\ x^t & 1 \end{bmatrix}$$

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- Finally, let $\mathbf{Z}$ be the matrix whose diagonal elements are $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{U}_k$ and $\mathbf{U}_k^-$.
- It is possible to map the coefficients of the objective function into a matrix $\mathbf{C}$, and likewise for the constraints into matrices $\mathbf{A}_s$ and vector $\mathbf{b}_s$ with $s = 1 : S$ where S is the total number of constraints.

Semidefinite Relaxations (1).

Using these, one can build a semidefinite relaxation based on the linear relaxation that has the following form:

$$\begin{aligned} \max_{\mathbf{Z}} \quad & \operatorname{trace}(\mathbf{CZ}) \\ \text{s.t.} \quad & \operatorname{trace}(\mathbf{A}_s\mathbf{Z}) \leq \mathbf{b}_s \\ & \mathbf{Z} \succeq 0 \end{aligned} \tag{16}$$

This includes a small number of additional constraints added to the linear relaxation obtained by multiplying the first stage capacity constraint by each x_i and $(1 - x_i)$ when no probability constraint is present, or by multiplying the contraints on y otherwise.

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- However, it uses sparsity in the matrices and saves memory.
- Unfortunately, even this way quicly reaches memory limits, and does not permit the use of more additional constraints.
- In particular, Sherali-Adams constraints are known to bring a good tightening of SDP relaxations, but requires adding a number of constraints polynomial in the size of our problem.

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- This allows the addition of Sherali-Adams constraints
- In order to be able to calculate such large models, we use `SBmethod`

Outline

SQKP with Recourse – No Probability Constraint

N	K	MIP	LP	Gap	SDP1	Gap	SDP2	Gap
5	5	418,37	480,31	14,80%	484,50	15,81%	457,78	9,42%
5	10	651,18	725,40	11,40%	715,66	9,90%	687,43	5,57%
5	20	494,98+	580,16	17,21%	572,55	15,67%	545,97	10,30%
10	5	2122,91	2334,37	9,96%	2276,31	7,23%	2209,10	4,06%
10	10	2556,99+	2724,16	6,54%	2691,41	5,26%	2641,76	3,32%
10	20	2790,65+	3099,41	11,06%	2973,57	6,55%	2911,26	4,32%
20	5	10148,49+	10731,20	5,74%	10448,51	2,96%	10438,36	2,86%
20	10	9843,45+	10413,55	5,79%	10141,64	3,03%	10070,52	2,31%
20	20	9392,40+	9960,75	6,05%	9655,65	2,80%	9621,03	2,43%

SQKP with Recourse – PC on First Stage

Small number of objects

N	K	L	MIP	LP	Gap	SDP1	Gap	SDP2	Gap
5	5	5	418,37	480,31	14,80%	485,05	15,94%	457,83	9,43%
5	5	10	511,25	575,23	12,51%	574,96	12,46%	556,70	8,89%
5	5	20	445,91	732,34	64,24%	736,31	65,13%	663,72	48,85%
5	10	5	651,18	722,94	11,02%	718,39	10,32%	687,69	5,61%
5	10	10	534,85	671,66	25,58%	668,73	25,03%	632,40	18,24%
5	10	20	524,13	768,37	46,60%	769,09	46,74%	688,52	31,36%
5	20	5	490,97+	580,17	18,17%	580,92	18,32%	544,69	10,94%
5	20	10	477,01+	673,74	41,24%	673,89	41,27%	625,03	31,03%
5	20	20	405,04	458,43	13,18%	450,76	11,29%	427,92	5,65%

SQKP with Recourse – PC on First Stage

Medium number of objects

N	K	L	MIP	LP	Gap	SDP1	Gap	SDP2	Gap
10	5	5	2117,87	2334,37	10,22%	2288,02	8,03%	2195,25	3,65%
10	5	10	2569,74	2885,89	12,30%	2869,37	11,66%	2773,87	7,94%
10	5	20	2102,09	2424,14	15,32%	2361,34	12,33%	2255,65	7,30%
10	10	5	2538,90+	2722,97	7,25%	2700,04	6,35%	2646,45	4,24%
10	10	10	2509,62	2871,47	14,42%	2854,71	13,75%	2697,10	7,47%
10	10	20	2368,12+	2706,86	14,30%	2682,80	13,29%	2537,70	7,16%
10	20	5	2793,87+	3104,31	11,11%	3073,72	10,02%	2914,36	4,31%
10	20	10	2273,40+	2650,11	16,57%	2616,94	15,11%	2497,59	9,86%
10	20	20	2300,81+	2661,47	15,68%	2634,72	14,51%	2513,20	9,23%

SQKP with Recourse – PC on First Stage

Large number of objects

N	K	L	MIP	LP	Gap	SDP1	Gap	SDP2	Gap
20	5	5	9891,38+	10736,57	8,54%	10629,55	7,46%	10366,16	4,80%
20	5	10	8669,30+	9550,47	10,16%	9354,55	7,90%	8997,66	3,79%
20	5	20	9227,87+	9759,95	5,77%	9605,69	4,09%	9441,77	2,32%
20	10	5	9597,74+	10416,86	8,53%	10257,40	6,87%	9925,79	3,42%
20	10	10	9840,00+	10699,73	8,74%	10482,37	6,53%	10153,15	3,18%
20	10	20	9153,10+	10097,78	10,32%	9909,93	8,27%	9473,63	3,50%
20	20	5	9162,78+	9962,35	8,73%	9804,19	7,00%	9572,43	4,47%
20	20	10	9227,47+	10098,36	9,44%	9980,94	8,17%	9742,73	5,58%
20	20	20	9554,39+	10301,83	7,82%	10141,54	6,15%	9942,30	4,06%

SQKP with Recourse – PC on Second Stage

Small number of objects

N	K	R	MIP	LP	Gap	SDP1	Gap	SDP2	Gap
5	5	5	387,13	498,72	28,82%	499,86	29,12%	451,00	16,50%
5	5	10	375,84	586,97	56,17%	558,51	48,60%	483,54	28,65%
5	5	20	269,29	569,12	111,34%	544,59	102,23%	463,73	72,21%
5	10	5	509,00	686,85	34,94%	671,40	31,90%	602,46	18,36%
5	10	10	401,55	644,79	60,57%	626,91	56,12%	541,83	34,94%
5	10	20	424,01	746,53	76,06%	Error		626,27	47,70%
5	20	5	350,04+	583,42	66,67%	562,72	60,76%	488,63	39,59%
5	20	10	330,89+	587,80	77,64%	Error		491,36	48,50%
5	20	20	324,70+	644,36	98,45%	Error		535,07	64,79%

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N	K	R	MIP	LP	Gap	SDP1	Gap	SDP2	Gap
10	5	5	1746,31+	2450,77	40,34%	2278,38	30,47%	2006,52	14,90%
10	5	10	1 471,30	2347,14	59,53%	2145,92	45,85%	1846,73	25,52%
10	5	20	1706,12+	2576,19	51,00%	2367,18	38,75%	2081,01	21,97%
10	10	5	1985,61+	2625,88	32,25%	2501,02	25,96%	2253,74	13,50%
10	10	10	1610,23+	2521,97	56,62%	2320,06	44,08%	2011,04	24,89%
10	10	20	1484,20+	2588,32	74,39%	Error		2062,23	38,95%
10	20	5	2337,37+	3126,34	33,75%	2938,07	25,70%	2708,16	15,86%
10	20	10	1473,93+	2330,17	58,09%	Error		1890,29	28,25%
10	20	20	1668,32+	2684,61	60,92%	Error		2295,64	37,60%

SQKP with Recourse – PC on Second Stage

Large number of objects

N	K	R	MIP	LP	Gap	SDP1	Gap	SDP2	Gap
20	5	5	8251,30+	10794,02	30,82%	9796,33	18,72%	9313,62	12,87%
20	5	10	6618,26+	10045,58	51,79%	8871,73	34,05%	7952,43	20,16%
20	5	20	6206,33+	10196,77	64,30%	8898,66	43,38%	7788,08	25,49%
20	10	5	7787,78+	10595,91	36,06%	9697,67	24,52%	8873,26	13,94%
20	10	10	7204,75+	10745,66	49,15%	9524,76	32,20%	8635,20	19,85%
20	10	20	6593,71+	10888,11	65,13%	Error		8704,24	32,01%
20	20	5	7316,29+	10128,16	38,43%	Error		8796,46	20,23%
20	20	10	7097,60+	10613,37	49,53%	Error		8903,78	25,45%
20	20	20	6142,95+	10276,54	67,29%	Error		8440,61	37,40%

SQKP with Recourse – PC on Both Stages

Small number of items

N	K	R	L	MIP	LP	Gap	SDP1	Gap	SDP2	Gap
5	5	10	10	463,46	834,46	80,05%	819,65	76,85%	693,89	49,72%
5	5	10	20	398,05	672,37	68,92%	667,77	67,76%	559,44	40,55%
5	5	20	10	440,40	755,80	71,62%	759,38	72,43%	675,50	53,38%
5	5	20	20	423,21	655,93	54,99%	652,25	54,12%	564,84	33,46%
5	10	10	10	418,37	611,58	46,18%	613,19	46,57%	548,50	31,11%
5	10	10	20	390,99	693,50	77,37%	684,86	75,16%	581,57	48,74%
5	10	20	10	363,23	713,09	96,32%	Error		598,49	64,77%
5	10	20	20	407,18	594,84	46,09%	Error		542,67	33,28%
5	20	10	10	396,17	625,25	57,82%	Error		518,08	30,77%
5	20	10	20	399,34+	693,73	73,72%	Error		592,39	48,34%
5	20	20	10	311,84+	608,15	95,02%	Error		492,17	57,83%
5	20	20	20	256,62+	579,57	125,85%	Error		467,36	82,12%

SQKP with Recourse – PC on Both Stages

Medium number of items

N	K	R	L	MIP	LP	Gap	SDP1	Gap	SDP2	Gap
10	5	10	10	1490,57	2309,03	54,91%	2067,71	38,72%	1791,97	20,22%
10	5	10	20	1906,60+	2658,03	39,41%	2582,91	35,47%	2296,32	20,44%
10	5	20	10	1705,78+	2674,55	56,79%	2452,71	43,79%	2150,11	26,05%
10	5	20	20	1644,76	2596,80	57,88%	2426,10	47,50%	2070,79	25,90%
10	10	10	10	2052,00+	3012,00	46,78%	2906,89	41,66%	2528,76	23,23%
10	10	10	20	1665,16+	2478,32	48,83%	2365,18	42,04%	2080,79	24,96%
10	10	20	10	1528,63+	2539,21	66,11%	Error		2039,77	33,44%
10	10	20	20	1533,78+	2677,80	74,59%	Error		2154,39	40,46%
10	20	10	10	1374,09+	2393,28	74,17%	Error		1919,10	39,66%
10	20	10	20	1756,33+	2450,39	39,52%	Error		2165,53	23,30%
10	20	20	10	1343,47+	2488,20	85,21%	Error		2022,24	50,52%
10	20	20	20	1431,71+	2461,02	71,89%	Error		1997,20	39,50%

SQKP with Recourse – PC on Both Stages

Large number of items

N	K	R	L	MIP	LP	Gap	SDP1	Gap	SDP2	Gap
20	5	10	10	7553,57+	11122,60	47,25%	10323,54	36,67%	9054,72	19,87%
20	5	10	20	7101,00+	10402,08	46,49%	9603,26	35,24%	8426,48	18,67%
20	5	20	10	6281,94+	10394,71	65,47%	9358,13	48,97%	7973,97	26,93%
20	5	20	20	6527,29+	10542,27	61,51%	9431,88	44,50%	8057,84	23,45%
20	10	10	10	6433,77+	10082,71	56,72%	Error		7874,86	22,40%
20	10	10	20	6735,73+	10428,89	54,83%	Error		8434,00	25,21%
20	10	20	10	6403,89+	10554,15	64,81%	Error		8220,27	28,36%
20	10	20	20	6443,44+	10568,39	64,02%	Error		8225,97	27,66%
20	20	10	10	6818,81+	10341,07	51,66%	Error		8596,85	26,08%
20	20	10	20	6727,75+	10458,95	55,46%	Error		8661,97	28,75%
20	20	20	10	6556,23+	10579,31	61,36%	Error		9121,42	39,13%
20	20	20	20	6166,59+	10361,01	68,02%	Error		9282,24	50,52%

Outline

Approximation

- We use two types of approximations

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  - Based on the result of the linear relaxation

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- We use two types of approximations
  - Based on the result of the linear relaxation
  - Based on the result of the second semidefinite relaxation
- We perform multiple iterations of randomized rounding and keep the best result
- We compare these to the optimum when known, and otherwise to the best result found by `CPLEX`

SQKP with Recourse – No probability Constraint.

N	K	MIP	RRSB	Gap	RRLP	Gap
5	5	418,37	418,37	0,00%	418,37	0,00%
5	10	651,18	650,53	0,10%	641,28	1,52%
5	20	494,98+	479,67	3,09%-	448,80	9,33%-
10	5	2122,91	2117,81	0,24%	1881,03	11,39%
10	10	2556,99+	2544,78	0,48%-	2433,86	4,82%-
10	20	2790,65+	2776,38	0,51%-	2497,71	10,50%-
20	5	10148,49+	10085,70	0,62%-	9817,29	3,26%-
20	10	9843,45+	9797,14	0,47%-	8840,82	10,19%-
20	20	9392,40+	9280,95	1,19%-	8046,96	14,32%-

SQKP with Recourse – PC on First Stage.

Small number of objects

N	K	L	MIP	RRSB	Gap	RRLP	Gap
5	5	5	418,37	418,37	0,00%	418,37	0,00%
5	5	10	511,25	511,25	0,00%	511,25	0,00%
5	5	20	445,91	445,91	0,00%	445,91	0,00%
5	10	5	651,18	650,53	0,10%	645,15	0,93%
5	10	10	534,85	534,70	0,03%	513,76	3,94%
5	10	20	524,13	521,19	0,56%	490,32	6,45%
5	20	5	490,97+	487,39	0,73%-	446,79	9,00%-
5	20	10	477,01+	471,97	1,06%-	445,12	6,69%-
5	20	20	405,04	400,32	1,17%	342,64	15,41%

SQKP with Recourse – PC on First Stage.

Medium number of objects							
N	K	L	MIP	RRSB	Gap	RRLP	Gap
10	5	5	2117,87	2103,60	0,67%	1886,00	10,95%
10	5	10	2569,74	2562,69	0,27%	2540,35	1,14%
10	5	20	2102,09	2102,09	0,00%	1969,17	6,32%
10	10	5	2538,90+	2525,76	0,52%-	2400,58	5,45%-
10	10	10	2509,62	2468,80	1,63%	2445,69	2,55%
10	10	20	2368,12+	2340,03	1,19%-	2138,22	9,71%-
10	20	5	2793,87+	2774,83	0,68%-	2502,73	10,42%-
10	20	10	2273,40+	2222,92	2,22%-	2027,14	10,83%-
10	20	20	2300,81+	2244,35	2,45%-	1998,29	13,15%-

SQKP with Recourse – PC on First Stage.

Large number of objects

N	K	L	MIP	RRSB	Gap	RRLP	Gap
20	5	5	9891,38+	9837,38	0,55%-	9550,74	3,44%-
20	5	10	8669,30+	8668,99	0,00%-	7112,14	17,96%-
20	5	20	9227,87+	9169,95	0,63%-	8177,94	11,38%-
20	10	5	9597,74+	9462,27	1,41%-	8559,57	10,82%-
20	10	10	9840,00+	9785,31	0,56%-	8329,95	15,35%-
20	10	20	9153,10+	9019,64	1,46%-	8359,73	8,67%-
20	20	5	9162,78+	9027,99	1,47%-	7830,69	14,54%-
20	20	10	9227,47+	9177,07	0,55%-	8291,68	10,14%-
20	20	20	9554,39+	9490,85	0,67%-	8249,59	13,66%-

SQKP with Recourse – PC on Second Stages

Small number of items

N	K	R	MIP	RRSB	Gap	RRLP	Gap
5	5	5	387,13	387,13	0,00%	364,28	5,90%
5	5	10	375,84	373,82	0,54%	359,89	4,24%
5	5	20	269,29	242,63	9,90%	235,89	12,40%
5	10	5	509,00	471,39	7,39%	455,65	10,48%
5	10	10	401,55	377,28	6,05%	363,37	9,51%
5	10	20	424,01	393,83	7,12%	390,78	7,84%
5	20	5	350,04+	326,24	6,80%-	314,07	10,28%-
5	20	10	330,89+	300,06	9,32%-	286,86	13,31%-
5	20	20	324,70+	283,71	12,62%-	257,40	20,73%-

SQKP with Recourse – PC on Second Stages

Medium number of items

N	K	R	MIP	RRSB	Gap	RRLP	Gap
10	5	5	1746,31+	1659,46	4,97%-	1520,37	12,94%-
10	5	10	1 471,30	1370,99	6,82%	1359,02	7,63%
10	5	20	1706,12+	1648,39	3,38%-	1500,41	12,06%-
10	10	5	1985,61+	1802,62	9,22%-	1662,11	16,29%-
10	10	10	1610,23+	1492,97	7,28%-	1393,28	13,47%-
10	10	20	1484,20+	1404,30	5,38%-	1343,23	9,50%-
10	20	5	2337,37+	2146,39	8,17%-	1828,20	21,78%-
10	20	10	1473,93+	1364,00	7,46%-	1244,60	15,56%-
10	20	20	1668,32+	1529,80	8,30%-	1304,96	21,78%-

SQKP with Recourse – PC on Second Stages

Large number of items

N	K	R	MIP	RRSB	Gap	RRLP	Gap
20	5	5	8251,30+	7842,79	4,95%-	6092,09	26,17%-
20	5	10	6618,26+	6199,13	6,33%-	5747,03	13,16%-
20	5	20	6206,33+	5830,31	6,06%-	5596,13	9,83%-
20	10	5	7787,78+	7241,47	7,01%-	6325,79	18,77%-
20	10	10	7204,75+	6373,00	11,54%-	5241,47	27,25%-
20	10	20	6593,71+	5924,10	10,16%-	5119,86	22,35%-
20	20	5	7316,29+	6413,14	12,34%-	5485,20	25,03%-
20	20	10	7097,60+	6343,38	10,63%-	5214,48	26,53%-
20	20	20	6142,95+	5567,18	9,37%-	5265,23	14,29%-

SQKP with Recourse – PC on Both Stages

Small number of items

N	K	R	L	MIP	RRSB	Gap	RRLP	Gap
5	5	10	10	463,46	456,76	1,45%	454,64	1,90%
5	5	10	20	398,05	385,10	3,25%	388,92	2,29%
5	5	20	10	440,40	428,71	2,65%	418,83	4,90%
5	5	20	20	423,21	388,11	8,29%	352,87	16,62%
5	10	10	10	418,37	404,55	3,30%	379,33	9,33%
5	10	10	20	390,99	358,95	8,19%	334,19	14,53%
5	10	20	10	363,23	339,71	6,47%	337,41	7,11%
5	10	20	20	407,18	387,78	4,76%	359,58	11,69%
5	20	10	10	396,17	342,53	13,54%	331,81	16,25%
5	20	10	20	399,34+	367,06	8,08%-	339,77	14,92%-
5	20	20	10	311,84+	258,23	17,19%-	245,34	21,32%-
5	20	20	20	256,62+	237,49	7,45%-	218,82	14,73%-

SQKP with Recourse – PC on Both Stages

Medium number of items

N	K	R	L	MIP	RRSB	Gap	RRLP	Gap
10	5	10	10	1490,57	1314,27	11,83%	1328,07	10,90%
10	5	10	20	1906,60+	1792,08	6,01%-	1621,70	14,94%-
10	5	20	10	1705,78+	1650,47	3,24%-	1475,70	13,49%-
10	5	20	20	1644,76	1552,88	5,59%	1519,75	7,60%
10	10	10	10	2052,00+	1902,70	7,28%-	1619,88	21,06%-
10	10	10	20	1665,16+	1567,13	5,89%-	1399,87	15,93%-
10	10	20	10	1528,63+	1395,15	8,73%-	1194,46	21,86%-
10	10	20	20	1533,78+	1377,97	10,16%-	1220,59	20,42%-
10	20	10	10	1374,09+	1299,00	5,46%-	1212,59	11,75%-
10	20	10	20	1756,33+	1582,64	9,89%-	1389,44	20,89%-
10	20	20	10	1343,47+	1223,82	8,91%-	1168,96	12,99%-
10	20	20	20	1431,71+	1373,46	4,07%-	1300,72	9,15%-

SQKP with Recourse – PC on Both Stages

Large number of items

N	K	R	L	MIP	RRSB	Gap	RRLP	Gap
20	5	10	10	7553,57+	7091,11	6,12%-	5898,11	21,92%-
20	5	10	20	7101,00+	6528,83	8,06%-	5496,67	22,59%-
20	5	20	10	6281,94+	5794,51	7,76%-	4874,30	22,41%-
20	5	20	20	6527,29+	6041,43	7,44%-	5725,03	12,29%-
20	10	10	10	6433,77+	5885,38	8,52%-	5488,73	14,69%-
20	10	10	20	6735,73+	6196,62	8,00%-	5594,71	16,94%-
20	10	20	10	6403,89+	5814,78	9,20%-	4887,79	23,67%-
20	10	20	20	6443,44+	5850,45	9,20%-	4909,90	23,80%-
20	20	10	10	6818,81+	6179,65	9,37%-	5533,38	18,85%-
20	20	10	20	6727,75+	5975,15	11,19%-	4680,47	30,43%-
20	20	20	10	6556,23+	5874,70	10,40%-	4888,10	25,44%-
20	20	20	20	6166,59+	5353,32	13,19%-	5258,38	14,73%-

Summary

- We proposed new models of quadratic knapsack with recourse.
- We give results of relaxations and approximations for these new models.
- Outlook
  - Study how existing methods for non-stochastic knapsacks could be adapted to this model.
  - Study models where deselection is not allowed.